

Inflationary universe in the presence of a minimal measurable length

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Abstract

In this paper, we will study the effect of having a minimum measurable length on inflationary cosmology. We will analyse the inflationary cosmology in the Jacobson approach. In this approach, gravity is viewed as an emergent thermodynamical phenomena. We will demonstrate that the existence of a minimum measurable length will modify the Friedmann equations in the Jacobson approach. We will use this modified Friedmann equation to analyse the effect of minimum measurable length scale on inflationary cosmology. This analysis will be performed using the Hamiltonian-Jacobi approach. We compare our results to recent data, and find that our model may agree with the recent data.

1 Introduction

It is known that a connection exists between the thermodynamics and gravity. This connection was first investigated in the works of Bardeen, Carter and Hawking [4]. In this work, it was suggested that an analogy exists between the laws of thermodynamics and gravitational physics. However, the existence of this connection was only established after the discovery of Hawking radiation [5]. The Hawking radiation comes from black holes, as black holes behave as hot bodies with a temperature proportional to the surface gravity of the black hole. The black holes also have an entropy associated with them, and this entropy

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is proportional to the area of the horizon [6]. Thus, the black hole physics establishes a connection between the laws of thermodynamics and gravitational physics.

Motivated by the relation between thermodynamics and gravity, it has been proposed that gravity is actually only an emergent thermodynamic phenomena. This formalism in which gravity is described as a emergent phenomena is called the Jacobson formalism [1]. In this formalism, the Einstein field equation can be derived from the first law of thermodynamics $dQ = TdS$. Thus, in this formalism Einstein field equation is viewed as an equation of state for the spacetime. It is possible to derive the Friedmann equations from the Clausius relation [2,3]. In this analysis, the entropy was taken to be proportional to the apparent horizon of FRW universe. Thus, if A is the area of the apparent horizon with temperature T , then the entropy S can be written as

$$S = \frac{A}{4G}, \quad (1.1)$$

$$T = \frac{1}{2\pi\tilde{r}_A} \quad (1.2)$$

It may be noted that even though the entropy is proportional to the area of the apparent horizon, the temperature is not proportional to the surface gravity κ . In fact, it is possible to express the surface gravity κ as

$$\kappa = -\frac{1}{\tilde{r}_A} \left(1 - \frac{\dot{\tilde{r}}_A}{2 H \tilde{r}_A} \right). \quad (1.3)$$

It has been possible to use the Misner-Sharp energy relation to express the dynamical Friedman equation in the form of the first law of thermodynamics [3]. However, for consistency of the results an extra work term had to be added to the original expression for the first law of thermodynamics. It has been demonstrated that the work density W can be expressed in terms of the energy density of the universe ρ and the pressure of matter in the universe p as

$$W = \frac{1}{2}(\rho - p). \quad (1.4)$$

Using this work function, the dynamical Friedman equations could be expressed as

$$dE = T dS + W dV. \quad (1.5)$$

It may be noted that the Hawking radiation is connected to the uncertainty principle in quantum mechanics [7–10]. This connection is established by modeling the black hole as a n -dimensional sphere whose radius is related to the Schwarzschild radius. The emitted particles from the black hole obey the uncertainty principle, as the Hawking radiation is a purely quantum mechanical process. This fact can be used to derive the thermodynamical properties of the black hole [8,9]. It is known that the temperature of a black hole increases as the size of the black hole decreases. This temperature tends to infinity as the mass size of the black hole approaches zero, and this in turn leads to a catastrophic evaporation of the black hole.

This is because in string theory it is not possible to probe the spacetime below string length scale, and so string theory comes equipped with a minimum measurable length

scale which is the string length scale. Furthermore, the energy required to probe the spacetime below Planck length is more than the energy required to form a mini black hole in that region of spacetime. Thus, there are strong indications that there exists a minimum measurable length scale for the spacetime [11–16]. In fact, the existence of a minimum measurable length scale is universal feature of almost all approaches to quantum gravity [11–13]. The existence of the minimum measurable length scale is not consistent with the usual uncertainty principle. This is because according to the usual uncertainty principle, it is possible to measure the length to an arbitrary accuracy, if the momentum is not measured. However, it is possible to generalize the usual uncertainty principle to a generalized uncertainty principle (GUP), such that this new uncertainty principle is consistent with the existence of a minimum measurable length.

As the Hawking radiation depends on the uncertainty principle, the modification of the usual uncertainty principle to the generalized uncertainty principle, also modifies the thermodynamics of the black holes. It has been demonstrated that there is no catastrophic evaporation of the black hole in this modified thermodynamics [7–10]. This is because a black hole remnant forms in the GUP deformed thermodynamics. Thus, the GUP prevents the black hole from evaporating completely, just like the standard uncertainty principle prevents the hydrogen atom from collapsing. In fact, the usual relation between the entropy and the area of a black hole gets significantly modified due to the GUP [7, 10].

As the Friedmann equations are viewed as thermodynamical relations in the Jacobson formalism, the modification of the thermodynamics by the GUP will also modify the Friedmann equations in the Jacobson formalism. Such modification to the Friedmann equations has been recently studied [17]. In this analysis, the relation between the area of the apparent horizon and entropy was used for calculating the modification to the Friedman equations from the GUP. It has been demonstrated that in these GUP modified Friedman equations, there exists a maximum energy density close to Planck density. Thus, the maximum energy density is reached in a finite time. This means that in this model there is an absence of the cosmological evolution beyond this point from a spacetime prospective. In fact, it was demonstrated that the maximum energy density and a general nonsingular evolution is independent of the equation of state and the spacial curvature k . This state of maximum energy density was reached in a finite time [21]. So, in this model, the big bang singularity is not accessible, and the energy density in the spacetime cannot be extended beyond Planck density. This bound on the energy density of spacetime occurs because of the GUP. As the Friedmann equations get modified due to the GUP, we expect the inflationary cosmology also to get modified due to the GUP in the thermodynamical approach. So, in this work we will study the inflationary cosmology using the Jacobian formalism. We will also study the effect of GUP deformed thermodynamics on the inflationary cosmology

It is possible to study the inflationary cosmology by using a potential to derive the properties of the inflationary cosmology. Thus, a specific form of the potential is chosen, and the model of the inflationary cosmology depends on the details of the potential chosen [18]. However, it is also possible to use the Hamilton-Jacobi formalism to model the inflationary universe [19]. In this formalism, the Hubble parameter is expressed in term of a scalar field. It is possible to deducing a form of the potential in the the Hamilton-Jacobi formalism. Furthermore, it is also possible to obtain an exact solution for the scalar field

in this approach. Thus, we will use the Hamilton-Jacobi formalism in this work.

The paper is organized as follows, in Sec. 2, we review the derivation of modified Friedmann equations due to GUP [17] in the context of the first law of thermodynamics. In Sec. 3, we discuss direct implications of the modified Friedmann equations on inflation. Finally, in Sec. 4, we summarize our results. We also discuss some possible extensions of this present work in this last section.

2 Friedmann Equations and Minimum Length

In this section, we briefly review the derivation of Friedman equations from the first law of thermodynamics relation with the apparent horizon of FRW universe with assuming that the entropy is proportional to the area of the apparent horizon [1–3]. We then review briefly the calculations of the impact of GUP on Friedman equations through this thermodynamic approach [17]. The $(n + 1)$ -dimensional FRW universe is represented by the following spacetime metric:

$$ds^2 = h_{ab}dx^a dx^b + \tilde{r}^2 d\Omega_{n-1}^2, \quad (2.1)$$

where $\tilde{r} = a(t)r$, $x^a = (t, r)$, $h_{ab} = (-1, a^2/(1 - kr^2))$, $d\Omega_{n-1}^2$ is the metric of $(n - 1)$ -dimensional sphere, $a, b = 0, 1$ and the spatial curvature constant k takes the values $0, 1, -1$ for a flat, closed and open universe, respectively. The apparent horizon is calculated from the relation $h^{ab}\partial_a\tilde{r}\partial_b\tilde{r} = 0$. So, its radius can be written as [2]:

$$\tilde{r}_A = a r = \frac{1}{\sqrt{H^2 + k/a^2}}, \quad (2.2)$$

where $H = \dot{a}/a$ is the Hubble parameter. We assume that the FRW universe is occupied with a perfect fluid. Now we can express the energy-momentum tensor

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}. \quad (2.3)$$

where u_μ is the four velocity of the fluid. The energy conservation law ($T^{\mu\nu}_{;\nu} = 0$) leads to the continuity equation

$$\dot{\rho} + nH(\rho + p) = 0. \quad (2.4)$$

Motivated by the arguments of [22], the work density W can be defined as follows

$$W = -\frac{1}{2}T^{ab}h_{ab}, \quad (2.5)$$

where T_{ab} is the projection of the energy-momentum tensor $T_{\mu\nu}$ in the normal direction. For our case, the work function is simplified to be

$$W = \frac{1}{2}(\rho - p). \quad (2.6)$$

Now, we briefly review the derivation of Einstein equations in the thermodynamic approach [3]. Thus, we will start by writing the first law of thermodynamics as

$$dE = T dS + W dV. \quad (2.7)$$

Utilizing Eq. (2.7), the infinitesimal change in the total energy ($E = \rho V$) could be calculated as following

$$\begin{aligned} dE &= \rho dV + V d\rho \\ &= n\Omega_n \tilde{r}_A^{n-1} \rho d\tilde{r}_A + \Omega_n \tilde{r}_A^n d\rho. \end{aligned} \quad (2.8)$$

Now the following result can be obtained,

$$dE = n\Omega_n \tilde{r}_A^{n-1} \rho d\tilde{r}_A - n\Omega_n \tilde{r}_A^n (\rho + p) H dt \quad (2.9)$$

Here we have used the continuity equation (2.4) and Eq. (2.8). For the work term $W dV$, we obtain the following expression

$$W dV = \frac{1}{2} n\Omega_n \tilde{r}_A^{n-1} (\rho - p) d\tilde{r}_A. \quad (2.10)$$

For the term $T dS$, we should use definition of Hawking temperature of Eq. (1.2) as well as the entropy-area law. The Hawking temperature can be obtained from the surface gravity as

$$T = \frac{\kappa}{2\pi}. \quad (2.11)$$

Here the surface gravity is calculated as

$$\begin{aligned} \kappa &= \frac{1}{2\sqrt{-h}} \partial_a (\sqrt{-h} h^{ab} \partial_b \tilde{r}) \\ &= -\frac{1}{\tilde{r}_A} \left(1 - \frac{\dot{\tilde{r}}_A}{2 H \tilde{r}_A} \right) \end{aligned} \quad (2.12)$$

We also use the entropy-area law of Eq. (1.1), and the expression for the area for a n -dimensional sphere $A = n\Omega_n \tilde{r}_A^{n-1}$, to obtain the following expression for the temperature

$$\begin{aligned} T dS &= \frac{\kappa}{2\pi} d \left(\frac{n\Omega_n \tilde{r}_A^{n-1}}{4G} \right) \\ &= -\frac{1}{2\pi \tilde{r}_A} \left[1 - \frac{\dot{\tilde{r}}_A}{2 H \tilde{r}_A} \right] \left(\frac{n(n-1)\Omega_n}{4G} \tilde{r}_A^{n-2} \right) \end{aligned} \quad (2.13)$$

Now by using Eqs. (2.9, 2.10 and 2.13), and the first law of thermodynamics given by Eq. (2.7), we obtain

$$\frac{d\tilde{r}_A}{\tilde{r}_A^3} = \frac{8\pi G}{n-1} (\rho + p) H dt. \quad (2.14)$$

We also use Eq. (2.2) to obtain $d\tilde{r}_A = -H\tilde{r}_A^3 \left(\dot{H} - k/a^2 \right) dt$ [2]. Thus, the Eq. (2.14) produces the dynamical Friedman equation.

$$\dot{H} - \frac{k}{a^2} = -\frac{8\pi G}{n-1}(\rho + p) \quad (2.15)$$

Furthermore, using the continuity equation (2.4) and integrating Eq. (2.15), one obtains the following result,

$$H^2 + \frac{k}{a^2} = \frac{8\pi G}{n(n-1)}\rho. \quad (2.16)$$

This is the Friedmann equation for $(n+1)$ -dimensional FRW universe.

Now, we review briefly the modification of the Friedman equations by the GUP. Thus, we will first review the GUP [11, 12], and its effect on the area-entropy law [7–10]. The existence of a minimum measurable length originates as an intriguing prediction of various frameworks of quantum gravity such as string theory [11] and black hole physics [12]. This implies a direct modification of the standard uncertainty principle [11–16]. Thus, according to GUP, the uncertainty in the length is modified to

$$\Delta x \geq \frac{\hbar}{\Delta p} \left[1 + \frac{\beta \ell_P^2}{\hbar^2} (\Delta p)^2 \right], \quad (2.17)$$

where ℓ_P is the Planck length and β is a dimensionless constant which is generated from some quantum gravitational effect. The new correction term in Eq. (2.17) becomes effective when the momentum and length scales are of order the Planck mass and of the Planck length, respectively. It has been demonstrated that the Eq.(2.17), implies the existence of minimal measurable length scale,

$$\Delta x \geq \Delta x_{min} = 2\beta \ell_P \quad (2.18)$$

As the black hole thermodynamics is related to the uncertainty principle, the GUP modifies the thermodynamics of the black hole. This modification of the thermodynamics of a black hole also modifies the Bekenstein-Hawking entropy of a black hole [7–10]). Thus, GUP modifies the relation between the entropy of a black hole and the area of its horizon, [7–10, 17])

$$\frac{dS}{dA} = \frac{\alpha}{8\ell_P^2} \frac{1}{A \left[1 - \sqrt{1 - \frac{\alpha}{A}} \right]}. \quad (2.19)$$

where $\alpha = 4\beta\ell_P^2\pi$, so the expression in Eq. (2.19) can be integrated to obtain the GUP modified entropy-area law [7–10, 17]).

$$S = \frac{1}{8\ell_P^2} \left[A + \sqrt{A^2 - A\alpha} - \frac{\alpha}{2} \ln \left(A + \sqrt{A^2 - A\alpha} - \frac{\alpha}{2} \right) \right] + S_0 \quad (2.20)$$

where S_0 is an integration constant. Thus, Eq. (2.20) is the modified Bekenstein-Hawking entropy. This modification has occurred due to the existence of a minimum measurable length scale or the GUP.

Now in the thermodynamical approach the Friedmann equation can be calculated from the entropy of the apparent horizon of FRW universe. As the entropy-area law is modified because of the GUP, we can expect that the Friedmann equations will also be modified because of GUP. In fact, it is possible to calculate the exact form of the GUP deformed Friedmann equations [17],

$$\begin{aligned} \frac{8\pi G}{3}(\rho - \Lambda) &= \frac{1}{2} \left(H^2 + \frac{k}{a^2} \right) \\ &+ \frac{4\pi}{3\alpha} \left[1 - \left(1 - \frac{\alpha}{4\pi} \left(H^2 + \frac{k}{a^2} \right) \right)^{\frac{3}{2}} \right], \end{aligned} \quad (2.21)$$

$$-4\pi G(\rho + p) = \left(\dot{H} - \frac{k}{a^2} \right) \frac{\alpha}{8\pi} \frac{\left(H^2 + \frac{k}{a^2} \right)}{\left[1 - \left(1 - \frac{\alpha}{4\pi} \left(H^2 + \frac{k}{a^2} \right) \right)^{\frac{1}{2}} \right]}. \quad (2.22)$$

Here the GUP modified Friedmann equations [17] have been calculated by using Eq. (2.21) and Eq. (2.22), in the the thermodynamical approach [3].

These consequences of this modified Friedmann equations have been studied [17]. It has been demonstrated that these equations lead to bounded energy density ρ as the inequality $H^2 + \frac{k}{a^2} \leq 4\pi/\alpha$ must be satisfied. This is because if this inequality is not satisfied, the density will become complex. By studying solutions to the modified Raychaudhuri equation [17], using phase space method [21], it was found that the big bang singularity is not accessible in this description. This was because the spacetime itself can not be extended beyond Planck density as a result of the GUP modified thermodynamic approach to gravity.

3 Inflation and Minimal Length

Now we can analyze the inflationary cosmology in the presence of a minimum measurable length. In this regards, the Hamilton-Jacobi approach is picked out as an appropriate formalism for studying inflation and its interesting prediction, quantum perturbations. We set $k = 0$ in our case for studying inflation. Considering the density given by $\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi)$, the modified Friedman equation will be as follows:

$$\frac{1}{2}H^2 + \frac{4\pi}{\alpha} \left[1 - \left(\frac{\alpha}{4\pi} H^2 \right)^{3/2} \right] = \frac{8\pi G}{3} \left(\frac{\dot{\phi}^2}{2} + V(\phi) \right) \quad (3.1)$$

Note that, from the Friedmann equation (3.1), it is realized that the expression $\alpha H^2/4\pi$ must be always smaller than unity. In the rest of the work, we take into account this and expand the term up to the third order.

The Hamilton-Jacobi formalism is utilized as a strong tool to provide numerous inflationary models with exactly known analytic solutions for the background expansion [23]. In this formalism, instead of potential, the Hubble parameter is introduced as a function of scalar field $H := H(\phi)$. Consequently, the time derivative of the Hubble parameter is reexpressed as $\dot{H} = \dot{\phi}H'$, where \prime denotes derivative with respect to scalar field. From the second

Friedmann equation (2.22), we have the time derivative of the scalar field as

$$\dot{\phi} = -\frac{1}{4\pi G} \frac{H'}{\left(1 + \frac{\alpha}{16\pi} H^2\right)}. \quad (3.2)$$

Substituting the relation in the first Friedmann equation comes to the Hamilton-Jacobi equation

$$\frac{1}{2}H^2 + \frac{4\pi}{\alpha} \left[1 - \left(\frac{\alpha}{4\pi} H^2\right)^{3/2}\right] = \frac{8\pi G}{3} \left(\frac{1}{2(4\pi G)^2} \frac{H'^2}{\left(1 + \frac{H}{4}\right)^2} + V(\phi)\right) \quad (3.3)$$

where the predicted potential of the model could easily be estimated, and the general behavior of the potential is investigated easily.

3.1 Perturbation

The most interesting aspect of the inflationary scenario is that the scenario predicts quantum perturbation in the very early times of the universe evolution. There are three types of perturbation known as scalar, vector, and tensor perturbations, and the most important ones are scalar and tensor perturbations. Scalar fluctuations become seeds for cosmic microwave background (CMB) anisotropies or for large scale structure (LSS) formation. Therefore, by measuring the spectra of the CMB anisotropies and density distribution, the corresponding primordial perturbations could be determined. Employing the amplitude of scalar perturbations from [36], the scalar spectra index is read as [37]

$$n_s - 1 = \frac{d \ln(\mathcal{P}_s)}{d \ln(k)} = 2\eta_H - 4\epsilon_H, \quad (3.4)$$

where ϵ_H and η_H are the first and second slow-rolling parameters, given by [37, 39]

$$\epsilon_H = -\frac{\dot{H}}{H^2}; \quad \eta_H = -\frac{|\ddot{\phi}|}{H|\dot{\phi}|}. \quad (3.5)$$

Besides scalar fluctuations, the inflationary scenario predicts tensor fluctuations, which is known as a gravitational wave, too. The produced tensor fluctuations induce a curved polarization in the CMB radiation and increase the overall amplitude of their anisotropies at a large scale. The physics of the early Universe could be specified by fitting the analytical results of CMB and density spectra to corresponding observational data. By contrast to scalar and vector perturbations, energy-momentum perturbations have no role in tensor perturbation equation. After doing some algebraic analysis, and obtaining the amplitude of tensor perturbations [36], the tensor spectra index is derived as [37]

$$n_T = \frac{d \ln(\mathcal{P}_T)}{d \ln(k)} = -2\epsilon_H. \quad (3.6)$$

The imprint of tensor fluctuations on the CMB bring this idea to indirectly determine its contribution to power spectra by measuring CMB polarization [38]. Such a contribution

could be expressed by the r quantity, which is known as tensor-to-scalar ratio and represents the relative amplitude of tensor-to-scalar fluctuations, $r = \mathcal{P}_T/\mathcal{P}_s$. Therefore, constraining r is one of the main goals of the modern CMB survey, and is given by

$$r = \frac{\mathcal{P}_T}{\mathcal{P}_s} = \frac{1}{2\pi G} \frac{H'^2}{H^2 \left(1 + \frac{\alpha}{16\pi} H^2\right)^2}. \quad (3.7)$$

Inflation lasts until the slow roll parameter ϵ_H approaches unity. Then, the final value of scalar field could be read from Eq.(3.5). In order to estimate the field value in the beginning of inflation, the common approach is to employ the number of e-folds equation. The number of e-folds, indicated by N , is expressed as following

$$N = \int_{\phi_i}^{\phi_e} \frac{H}{\dot{\phi}} d\phi, \quad (3.8)$$

where the subscripts " i " and " e " respectively denote the initial and end of inflation. Integrating the equation for specific Hubble parameter, and using $\epsilon_H = 1$, the initial value of scalar field could be calculated, which will be done in the following subsection.

3.2 Attractor Behavior

One of the absorbing aspect of the Hamilton-Jacobi formalism is that it allows to simply consider the attractor behavior of the model. The common approach is to suppose a homogeneous perturbation for Hubble parameter as $H = H_0 + \delta H$ [40], and insert it in the Hamiltonian-Jacobi equation up to the first order. If the expression $\delta H(\phi)/H_0(\phi)$ approaches to zero with increasing time, the attractor condition could be satisfied. In the following lines, the calculation is explained step by step for more clarity

$$\begin{aligned} \frac{H^2}{2} &\rightarrow \frac{1}{2}(H_0 + \delta H)^2 \\ &= \frac{1}{2}H_0^2 \left(1 + 2\frac{\delta H}{H_0}\right) \\ &= \frac{1}{2}H_0^2 + H_0\delta H \end{aligned} \quad (3.9)$$

Thus, we can write

$$\begin{aligned}
1 - \left(1 - \frac{\alpha}{4\pi} H^2\right)^{3/2} &= 1 - (1 - H)^{3/2} \\
&= 1 - \left(1 - \frac{3}{2}H + \frac{3}{8}H^2\right) \\
&= \frac{3}{2}\left(H - \frac{1}{4}H^2\right) \\
&= \frac{3}{2} \frac{\alpha}{4\pi} H_0^2 \left(1 + \frac{2\delta H}{H_0}\right) - \frac{3}{8} \left(\frac{\alpha}{4\pi}\right)^2 H_0^4 \left(1 + \frac{4\delta H}{H_0}\right) \\
&= \left(\frac{3}{2} \frac{\alpha}{4\pi} H_0^2 - \frac{3}{8} \left(\frac{\alpha}{4\pi}\right)^2 H_0^4\right) \\
&\quad + \left(\frac{3}{2} \left(\frac{\alpha}{4\pi}\right)^2 H_0 - \frac{3}{2} \left(\frac{\alpha}{4\pi}\right)^2 H^3\right) \delta H
\end{aligned} \tag{3.10}$$

Now we can write

$$\begin{aligned}
\frac{H'^2}{\left(1 + \frac{H}{2} + \frac{H^2}{16}\right)} &\rightarrow \frac{H'^2}{\left(1 + \frac{H}{2} + \frac{H^2}{16}\right)} \\
&= \frac{H_0^2 \left(1 + \frac{2\delta H'}{H_0}\right)}{1 + \frac{\alpha}{8\pi} H_0^2 \left(1 + \frac{2\delta H_0}{H_0}\right) + \frac{1}{16} \frac{\alpha^2}{(4\pi)^2} H_0^4 \left(1 + \frac{4\delta H}{H_0}\right)} \\
&= \frac{H_0'^2 + 2H_0'\delta H_0'}{A_1 + A_2},
\end{aligned} \tag{3.11}$$

where

$$\begin{aligned}
A_1 &= 1 + \frac{\alpha}{8\pi} H_0^2 + \frac{1}{16} \frac{\alpha^2}{(4\pi)^2} H_0^4 \\
A_2 &= \left(\frac{\alpha}{4\pi} H_0 + \frac{1}{4} \frac{\alpha^2}{(4\pi)^2} H_0^3\right) \delta H_0
\end{aligned} \tag{3.12}$$

So, we can write

$$\begin{aligned}
\frac{H'^2}{\left(1 + \frac{H}{2} + \frac{H^2}{16}\right)} &= \frac{H'^2}{A_1 + A_2 \delta H} + \frac{2H'\delta H'}{A_1 + A_2 \delta H} \\
&= \frac{H'^2}{A_1} \left(1 - \frac{A_2 \delta H}{A_1}\right) + \frac{H'\delta H'}{A_1} \left(1 - \frac{A_2}{A_1} \delta H\right)
\end{aligned} \tag{3.13}$$

To the first order, we can write

$$(H_0 \delta H) + \frac{4\pi}{\alpha} \left(3 \frac{\alpha}{4\pi} - \frac{3}{2} \frac{\alpha^2}{(4\pi)^2} H_0^3\right) \delta H = \frac{8\pi G}{3} \frac{1}{2(4\pi G)^2} \left[\frac{-H'^2 A_2 \delta H}{A_1^2} - \frac{H'\delta H'}{A_1} \right] \tag{3.14}$$

Thus, we can write

$$\left[H_0 + 3H_0 - \frac{3}{2}H_0^3 + \frac{1}{126\pi G} \frac{H_0'^2 A_2}{A_1^2}\right] \delta H = -\frac{H_0'}{A_1} \delta H' \frac{1}{12\pi G} \tag{3.15}$$

and

$$\begin{aligned}\mathbb{H}(\phi) &= \frac{\delta H'}{\delta H} \\ &= \frac{12\pi G A_1}{H'_0} \left[4H_0 - \frac{3\alpha}{8\pi} H_0^3 + \frac{1}{6\pi G} \frac{H'_0 A_2}{A_1^2} \right]\end{aligned}\quad (3.16)$$

Now we have

$$\delta H(\phi) = \delta H_0 \exp \int \mathbb{H}(\phi) d\phi. \quad (3.17)$$

The attractor behavior is satisfied if the above expression goes to zero by passing time.

3.3 Potential

The general form of the potential was introduced in Eq.(3.3). In order to study the behavior of the potential during the inflationary times, the potential could be drew in term of scalar field. Rewriting the Hamilton-Jacobi equation (3.3), the potential could be expressed by

$$V(\phi) = \frac{3}{8\pi G} \left[\frac{H^2}{2} - \frac{4\phi}{\alpha} \left(1 - \left(1 - \frac{\alpha}{4\pi} H^2 \right)^{3/2} \right) \right] - \frac{1}{2(4\pi G)^2} \frac{H'^2}{\left(1 + \frac{\alpha}{16\pi} H^2 \right)^2} \quad (3.18)$$

This a the general form of the potential in term of the Hubble parameter. To investigate the behavior of the potential, we need to introduce a function for the Hubble parameter in term on scalar field, that is what we are going to do in next subsection.

3.4 Typical Example

To go further, and investigate the result in more clear detail, it is necessary to propose a specific function for the Hubble parameter in term of scalar field. In this regards, we assume that there is $H = \mathcal{H}_1 \phi$, in which \mathcal{H}_1 is a constant. Substituting the definition in the above equations, the situation will be studied in more detail.

Inflation ends when the slow rolling parameter ϵ_H approaches unity. The general form of the equations were acquired in the previous section. In this section, we are going straight to the result for three different case, and consider the general behavior of tensor-to-scalar ratio, potential, and equation of state parameter ω (it is assumed that there is an equation of state as $p = \omega\rho$).

First Case: The tensor-to-scalar, the potential, and the equation of state parameter respectively have been plotted for $\beta = 9 \times 10^{-4}$, $\beta \mathcal{H}_1^2 = 1 \times 10^{-14}$, and three different values of number of e-folds as: $N = 55$ (solid line), 60 (dashed line) and 65 (dotted-dashed line). In this case, we see that all three ones are same as each others.

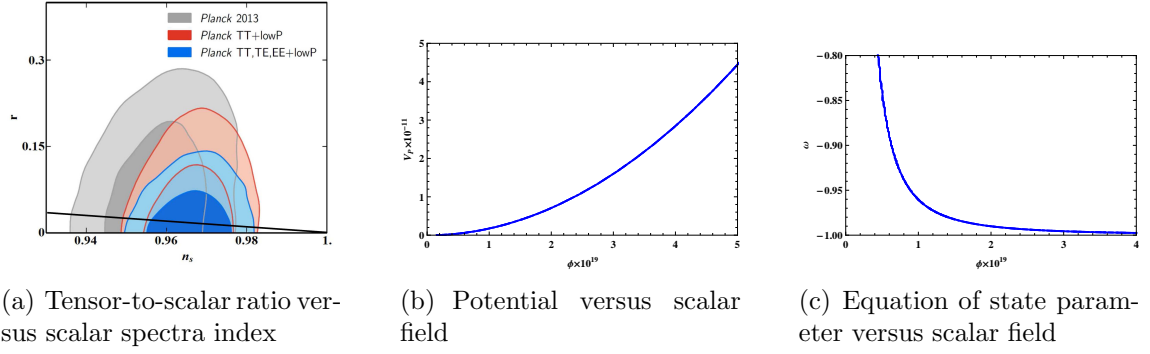


Figure 1: First Case

| N | 55 | 60 | 65 |
|----------|--------------------------|--------------------------|--------------------------|
| ϕ_e | 3.63886×10^{18} | 3.63886×10^{18} | 3.63886×10^{18} |
| ϕ_i | 3.62062×10^{19} | 3.78162×10^{19} | 3.93603×10^{19} |
| n_s | 0.963799 | 0.966816 | 0.969369 |
| P_s | 3.43628×10^{-9} | 3.08945×10^{-9} | 3.79943×10^{-9} |
| n_T | -0.0181003 | -0.0165919 | -0.0153156 |
| r | 0.0962005 | 0.0331838 | 0.0306312 |

Table 1: First Case

Second Case: The tensor-to-scalar, the potential, and the equation of state parameter respectively have been plotted for $\beta = 5 \times 10^{-4}$, $N = 60$, and three different values of number of e-folds as: $\beta \mathcal{H}_1^2 = 5 \times 10^{-15}$ (solid line), 7×10^{-15} (dashed line) and 9×10^{-15} (dotted-dashed line). In this case, we see that the plots, except the potentials, are same as each others.

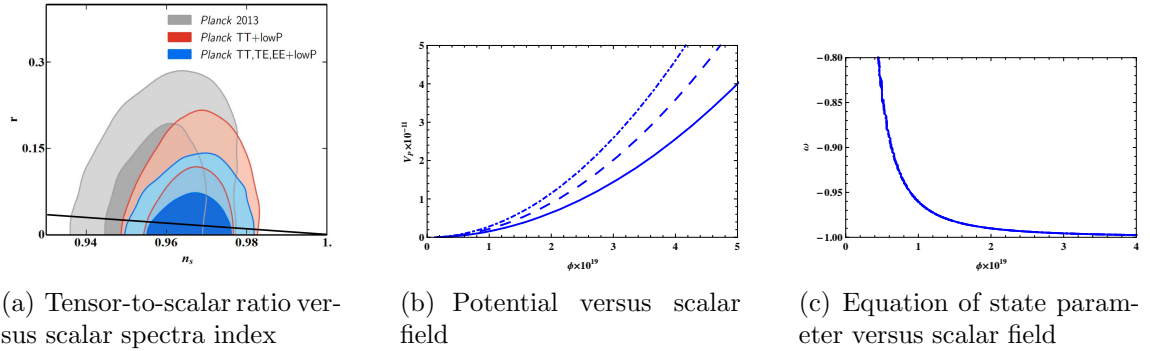


Figure 2: Second Case

Third Case: The tensor-to-scalar, the potential, and the equation of state parameter respectively have been plotted for $\mathcal{H}_1 = 3 \times 10^{-6}$, $N = 60$, and three different values of number of e-folds as: $\beta = 5 \times 10^{-4}$ (solid line), 7×10^{-4} (dashed line) and 9×10^{-4} (dotted-dashed line). In this case, we see that all three ones are same as each others.

| $\beta\mathcal{H}_1^2$ | 5×10^{-15} | 7×10^{-15} | 9×10^{-15} |
|------------------------|--------------------------|--------------------------|--------------------------|
| ϕ_e | 3.63886×10^{18} | 3.0754×10^{18} | 2.71225×10^{18} |
| ϕ_i | 3.78162×10^{19} | 3.79161×10^{19} | 3.77773×10^{19} |
| n_s | 0.966816 | 0.966991 | 0.966748 |
| P_s | 3.68051×10^{-9} | 5.20738×10^{-9} | 6.59768×10^{-9} |
| n_T | -0.0165619 | -0.0165046 | -0.0166261 |
| r | 0.0331838 | 0.0330092 | 0.0332522 |

Table 2: Second Case

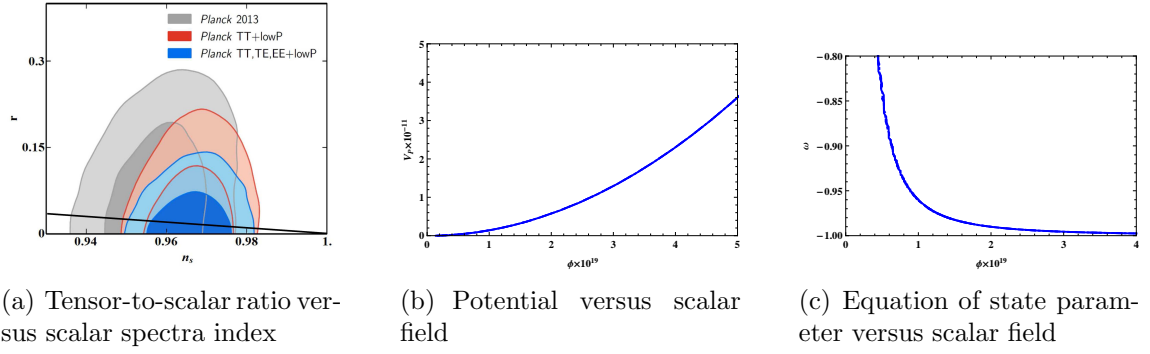


Figure 3: Third Case

| β | 5×10^{-15} | 7×10^{-15} | 9×10^{-15} |
|----------|--------------------------|--------------------------|--------------------------|
| ϕ_e | 3.8357×10^{18} | 3.24176×10^{18} | 2.85896×10^{18} |
| ϕ_i | 3.79715×10^{19} | 3.78051×10^{19} | 3.77123×10^{19} |
| n_s | 0.967087 | 0.966797 | 0.966633 |
| P_s | 3.36721×10^{-9} | 3.30856×10^{-9} | 3.27621×10^{-9} |
| n_T | -0.0164565 | -0.0166017 | -0.0166834 |
| r | 0.0329129 | 0.0332033 | 0.0333669 |

Table 3: Third Case

In all above three cases, we observe the same behavior for r , $V(\phi)$, and ω . The result displays that, the predicted values for tensor-to-scalar ratio is in good agreement with the latest observational data of Planck as $r < 0.10$. In addition, the general behavior of the potential sounds to be suitable. At the beginning of inflation, the scalar field (inflaton) stands on top of the potential, and it rolls down to the minimum with increasing time, or with reducing scalar field. Then, it is realized that the inflation in this work could be counted as a member of "*Large Field Model*" class of inflation. The third part of figures depicts the equation of state parameter in term of scalar field. It is exhibited that in the initial times of inflation, the parameter is near to -1 . Then, the Universe undergoes a quasi-de Sitter expansion for the Universe; a result that was expected. Finally, some parameters, such as amplitude of scalar perturbation, scalar spectra index, and tensor spectra index, have been

estimated and presented in Table.1, 2, and 3 for each case. The model prediction for the parameters are consistence with the Planck data, where $\ln(10^{10}\mathcal{P}_s) = 3.094 \pm 0.034$ and $n_s = 0.9645 \pm 0.0049$. It states that the model could be taken into account as a proper model for studying inflation.

Beside these, integrating $\int \mathbb{H}(\phi)d\phi$ during the inflation times interval shows that the expression is equal to a large negative value. Consequently, from Eq.(3.17), one could realized that the parameter δH approaches zero with increasing time, and the attractor behavior could be satisfied by the model.

4 Conclusions

In this paper, we have analyzed the effect of having a minimum measurable length scale on inflationary cosmology using inflationary cosmology in the Jacobian approach. This approach is motivated by the relation between gravity and thermodynamics where Einstein field equations are the equation of state for the geometry of spacetime. So, in this paper, the Friedmann equations are viewed as the Clausius relation. Then we analyze the modification of the Friedmann equations because of the existence of a minimum measurable length. This is done by analyzing the effect of the existence of a minimum measurable length on the entropy of the cosmological Horizon. Finally, these modified Friedmann equations are used for calculating the modifications to the inflationary cosmology. This analysis is done using the Hamiltonian-Jacobi approach. We thus explicitly calculate the effect of having a minimum measurable length scale on inflationary cosmology.

It may be noted that property of the spectrum of large-scale magnetic fields which is generated due to the breaking of the conformal invariance has been studied in the context of inflationary cosmology [29]. In this analysis, it has been demonstrated that the spectrum of the magnetic fields should not be perfectly scale-invariant. In fact, it was observed that this spectrum should be slightly red so that the amplitude of large-scale magnetic fields can be stronger than a certain value. In this analysis, it was assumed that the absence of amplification occurs due to the late-time action of some dynamo mechanism. It would be interesting to repeat this analysis by assuming the existence of a minimum measurable length scale. The inflation has also been studied using a non-canonical Lagrangian [30]-[31]. In this case, the modification to the kinetic term is modified. This modification to the kinetic term depends only on the fields and not the derivatives of the fields. The non-canonical inflation has also been studied in the context of string-inspired inflation models [32] The DBI action has also been used to study such models of inflation [33]-[34]. It has been demonstrated that the standard Hubble slow roll expansion to the non-canonical case can be generalized [35]. This generalization corresponds to the derivation of the expressions for observables in terms of the generalized slow roll parameters. It would be interesting to analyze the models with non-canonical kinetic term in the thermodynamic approach and also analyze the modification to these models that can occur because of the existence of a minimum measurable length scale.

Utilizing Hamilton-Jacobi formalism in the introduced generalized Friedmann equation allows one to properly study inflation epoch of the Universe evolution. The calculated result demonstrates that the model could suitably describe inflation, and it sounds that

the result is in consistence with the latest observational data of Planck. The tensor-to-scalar ratio is plotted versus n_s for three different cases, and in all of them the parameter r stands in the predicted bound by Planck as $r < 0.10$. The estimated values for n_s and \mathcal{P}_s indicate that these parameters are almost about $n_s = 0.9669$ and $\mathcal{P}_s = 3.620 \times 10^9$; which are in perfect agreement with Planck data, where $n_s = 0.9645 \pm 0.0049$ and $\mathcal{P}_s = 2.206 \times 10^9$. Considering the predicted potential behavior exhibits that the proposed model stands in *Large Field Class* of inflationary models, so that the scalar field leaves the top of the potential at the beginning of inflation and slowly rolls down to the minimum. In addition, behavior of equation of state parameter ω is in agreement with whole work. It expresses a quasi-de Sitter expansion for the universe, in which at the initial of inflation, the parameter is so close to -1 . Finally, investigation of attractor behavior for the model comes to another pleasant outcome. The homogenous perturbation δH approaches zero with increasing time for all three cases describing that the model could satisfy the attractor behavior.

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